Analysis of Nonlinear Dynamics by Square Matrix Method

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NOCE, Arcidosso, Sep. 2017

Write one turn map of Taylor expansion as square matrix

L.H. Yu, PRAB, 20, 034001 (2017)

Simplest example of nonlinear map :

$$x = x_0 \cos\mu + p_0 \sin\mu + \epsilon x_0^2 \sin\mu$$
$$p = -x_0 \sin\mu + p_0 \cos\mu + \epsilon x_0^2 \cos\mu,$$

Use $z = x - ip \text{ and } z^* = x + ip$

$$\begin{aligned}
z &= e^{i\mu} z_0 - \frac{i}{4}\epsilon e^{i\mu} z_0^2 - \frac{i}{2}\epsilon e^{i\mu} z_0 z_0^* - \frac{i}{4}\epsilon e^{i\mu} z_0^{*2} \\
z^* &= e^{-i\mu} z_0^* + \frac{i}{4}\epsilon e^{-i\mu} z_0^2 + \frac{i}{2}\epsilon e^{-i\mu} z_0 z_0^* + \frac{i}{4}\epsilon e^{-i\mu} z_0^{*2} \\
z^2 &= e^{2i\mu} z_0^2 - \frac{i}{2}\epsilon e^{2i\mu} z_0^3 - i\epsilon e^{2i\mu} z_0^2 z_0^* - \frac{i}{2}\epsilon e^{2i\mu} z_0 z_0^{*2} \\
zz^* &= z_0 z_0^* + \frac{i}{4}\epsilon z_0^3 + \frac{i}{4}\epsilon z_0^2 z_0^* - \frac{i}{4}\epsilon z_0 z_0^{*2} - \frac{i}{4}\epsilon z_0^{*3} \\
z^{*2} &= e^{-2i\mu} z_0^{*2} + \frac{i\epsilon}{2} e^{-2i\mu} z_0^2 z_0^* + i\epsilon e^{-2i\mu} z_0 z_0^{*2} + \frac{i\epsilon}{2} e^{-2i\mu} z_0^{*3} \\
z^3 &= e^{3i\mu} z_0^3 \\
&\dots \\
z^{*3} &= e^{-3i\mu} z_0^{*3}
\end{aligned},$$

$$\tilde{Z} = (1, z, z^*, z^2, zz^*, z^{*2}, z^3, z^2 z^*, zz^{*2}, z^{*3}). \longrightarrow \qquad Z = MZ_0,$$

At 3rd order M is 10x10 matrix:

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & M_{11} & M_{12} & M_{13} \\ 0 & 0 & M_{22} & M_{23} \\ 0 & 0 & 0 & M_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{bmatrix}, M_{22} = \begin{bmatrix} e^{2i\mu} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2i\mu} \end{bmatrix}, \ldots$$

$$M_{23} = \begin{bmatrix} -\frac{i}{2}\epsilon e^{2i\mu} & -i\epsilon e^{2i\mu} & -\frac{i}{2}\epsilon e^{2i\mu} & 0 \\ \frac{i}{4}\epsilon & \frac{i}{4}\epsilon & -\frac{i}{4}\epsilon & -\frac{i}{4}\epsilon \\ 0 & \frac{i}{2}\epsilon e^{-2i\mu} & i\epsilon e^{-2i\mu} & \frac{i}{2}\epsilon e^{-2i\mu} \end{bmatrix}$$

M is upper-triangular matrix with diagonal elements precisely known (the eigenvalues)

$$1, \{e^{i\mu}, e^{-i\mu}\}, \{e^{2i\mu}, 1, e^{-2i\mu}\}, \{e^{3i\mu}, e^{i\mu}, e^{-i\mu}, e^{-3i\mu}\}$$

-->2 eigenvectors

- Invariant subspace of eigenvalue $e^{i\mu}$ of dimension 2.
- In 3rd order, nonlinear dynamics is represented by a rotation in this 2 dimensional space
  10x10→2x2
- For higher order, the dimension of the invariant subspace is always much smaller than original dimension.
- Example, 7'th order , for 4 variables x, p_x, y, p_y
  330x330→4x4

We find left eigenvectors U, such that with Jordan matrix τ

$$UM = e^{i\mu I+\tau}U$$

$$U = \begin{bmatrix} u_1 \\ u_2 \\ \dots \\ \dots \\ u_{m-1} \end{bmatrix} \qquad \tau = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

There is **only one step to high order**, without iteration from low order to high order

- Example, 3'th order , for 2 variables
  U: 2x10 matrix M: 10x 10 τ: 2x2 matrix
- Example, 7'th order , for 4 variables
  U: 4x330 matrix M: 330x 330 τ : 4x4 matrix

$$UZ = UMZ_0 = e^{i\mu I+\tau}UZ_0.$$

Let $W \equiv UZ = \begin{bmatrix} w_0 \\ w_1 \\ \dots \\ w_{m-1} \end{bmatrix}$

$$\left.\begin{aligned} W &\equiv UZ \\ W_0 &\equiv UZ_0 \end{aligned}\right. \longrightarrow W = e^{i\mu I+\tau}W_0.$$

KAM theory states that the invariant tori are stable under small perturbation
There is a stable frequency, hence

$$W = e^{i\mu I+\tau} W_0 \cong e^{i(\mu+\phi)} W_0.$$

(I here is the identity matrix)

Compare left with right side

$$\tau W_0 \cong i\phi W_0.$$

So we must have an approximate "Coherent state":

$$\tau \begin{bmatrix} w_0 \\ w_1 \\ \dots \\ w_{m-1} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ 0 \end{bmatrix} \cong \begin{bmatrix} i\phi w_0 \\ i\phi w_1 \\ \dots \\ i\phi w_{m-1} \end{bmatrix} \longrightarrow i\phi = \frac{w_1}{w_0} \cong \frac{w_2}{w_1} \cong \frac{w_3}{w_2} \dots \frac{w_{m-1}}{w_{m-2}}$$

(Last row is very small, so it is still approximately correct)

Amplitude dependent tune ϕ

Action $|w_0|$ **is nearly a constant**

The Pendulum Equation as an 3rd order example

$$H = \frac{p^2}{2} + 1 - \cos(x) \qquad H = \frac{p^2}{2} + \frac{x^2}{2} - \frac{x^4}{24}$$

Expand Hamiltonian to 4'th order

$$z \equiv x - ip, \qquad z^* \equiv z + ip$$

$$\begin{aligned} \dot{z} &= iz - \frac{iz^3}{48} - \frac{1}{16} iz^2 z^* - \frac{1}{16} izz^{*2} - \frac{iz^{*3}}{48} \\ \dot{z^*} &= -iz^* + \frac{iz^{*3}}{48} + \frac{1}{16} iz^* z^{*2} + \frac{1}{16} iz^2 z^* + \frac{iz^3}{48} \end{aligned}$$

$$\frac{dz^2}{dt} \approx 2iz^2, \quad \frac{dzz^*}{dt} \approx 0, \quad \frac{dz^{*2}}{dt} \approx 2iz^{*2}, \quad \cdots \quad , \frac{dz^{*3}}{dt} \approx -3iz^{*3}$$

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & -\frac{i}{48} & -\frac{i}{16} & -\frac{i}{16} & -\frac{i}{48} \\ 0 & 0 & -i & 0 & 0 & 0 & \frac{i}{48} & \frac{i}{16} & \frac{i}{16} & \frac{i}{48} \\ 0 & 0 & 0 & 2i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3i \end{pmatrix}$$

$$U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{96} & 0 & -\frac{1}{32} & -\frac{1}{192} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{16} & 0 & 0 \end{pmatrix}$$

Left eigenvectors

$$\dot{Z} = MZ, \quad UM = (i\omega_0 I + \tau)U, \quad \tau \equiv \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$$W = UZ = \begin{bmatrix} u_0 Z \\ u_1 Z \end{bmatrix} \equiv \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}.$$

$$\dot{W} = U\dot{Z} = UMZ = (i\omega_0 I + \tau)UZ = (i\omega_0 I + \tau)W$$

$$\dot{w}_0 = i\omega_0 w_0 + w_1 = (i\omega_0 + \frac{w_1}{w_0})w_0 \equiv i(\omega_0 + \phi)w_0$$
$$\dot{w}_1 = i\omega_0 w_1$$

$$w_0 = u_0 Z = z + \frac{z^3}{96} - \frac{zz^{*2}}{32} - \frac{z^{*3}}{192} \approx z$$
$$w_1 = u_1 Z = -\frac{1}{16} i z^2 z^*$$

Lowest order approximation

$$\Delta\omega = \phi = -i\frac{w_1}{w_0} \cong -\frac{1}{16} zz^*$$

Frequency shift

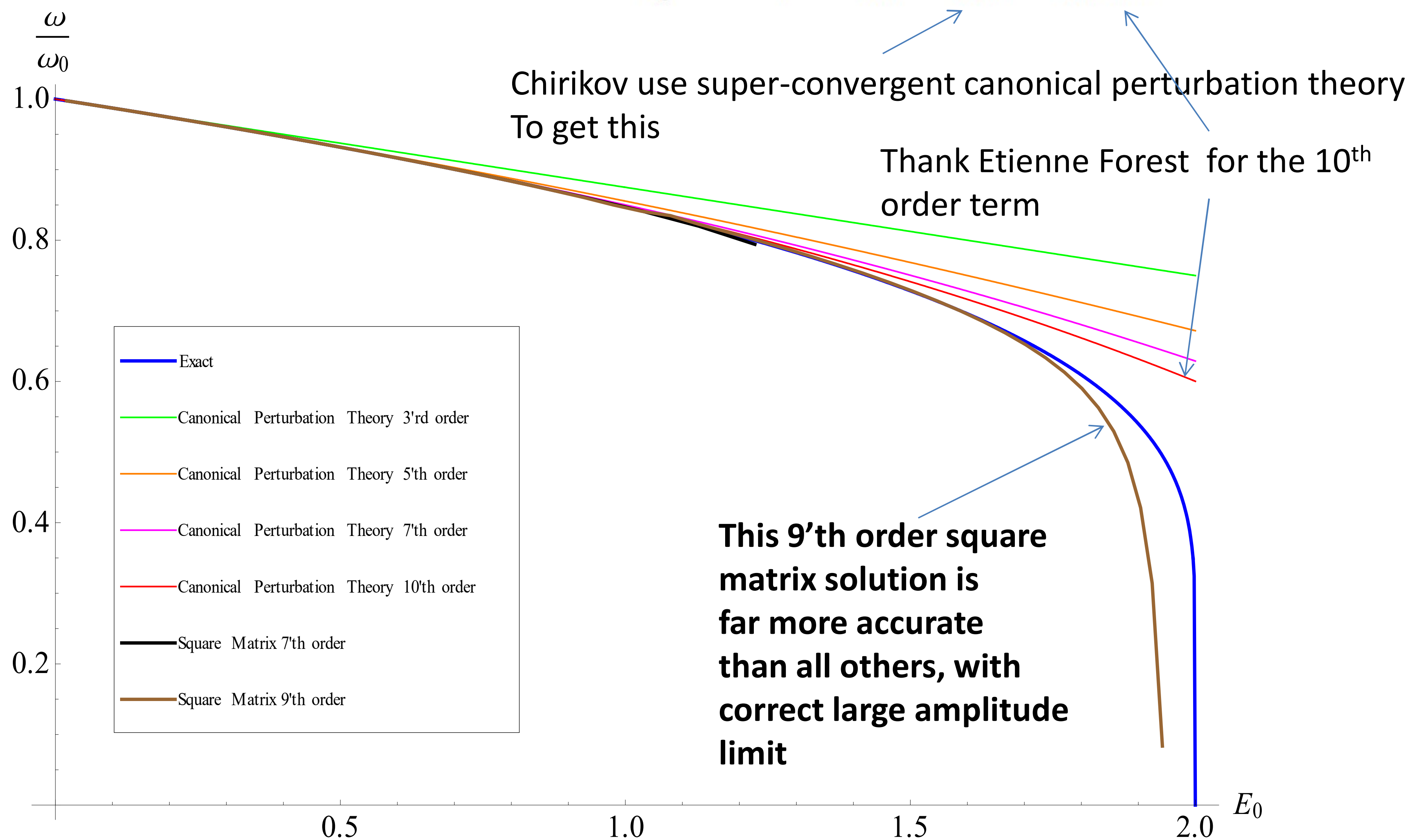
$E_0 = \frac{z^2}{2}$, $\omega_0 = 1$, and initially, $z^* = z = x_0$, so

$$\frac{\omega}{\omega_0} = \frac{\omega_0 + \Delta\omega}{\omega_0} \cong 1 - \frac{1}{8}E_0$$

A well known result

This is at 3'rd order. At 10'th order,

$$\frac{\omega}{\omega_0} = 1 - \frac{E_0}{8} - \frac{5E_0^2}{256} - \frac{11E_0^3}{2048} - \frac{469E_0^4}{262144}$$



Compare canonical perturbation theory with Square Matrix

Multi-turns

$$W(n) = e^{in\mu} e^{n\tau} W_0$$

Further derivation leads to

$$w_0(n) = e^{in\mu + in\phi + \frac{n^2}{2}\Delta + \cdots} w_0(n=0)$$

$$\Delta \equiv \frac{w_2}{w_0} - (\frac{w_1}{w_0})^2 \quad \approx 0$$

Gives **frequency fluctuation**, seems to be related to Liapunov exponent.

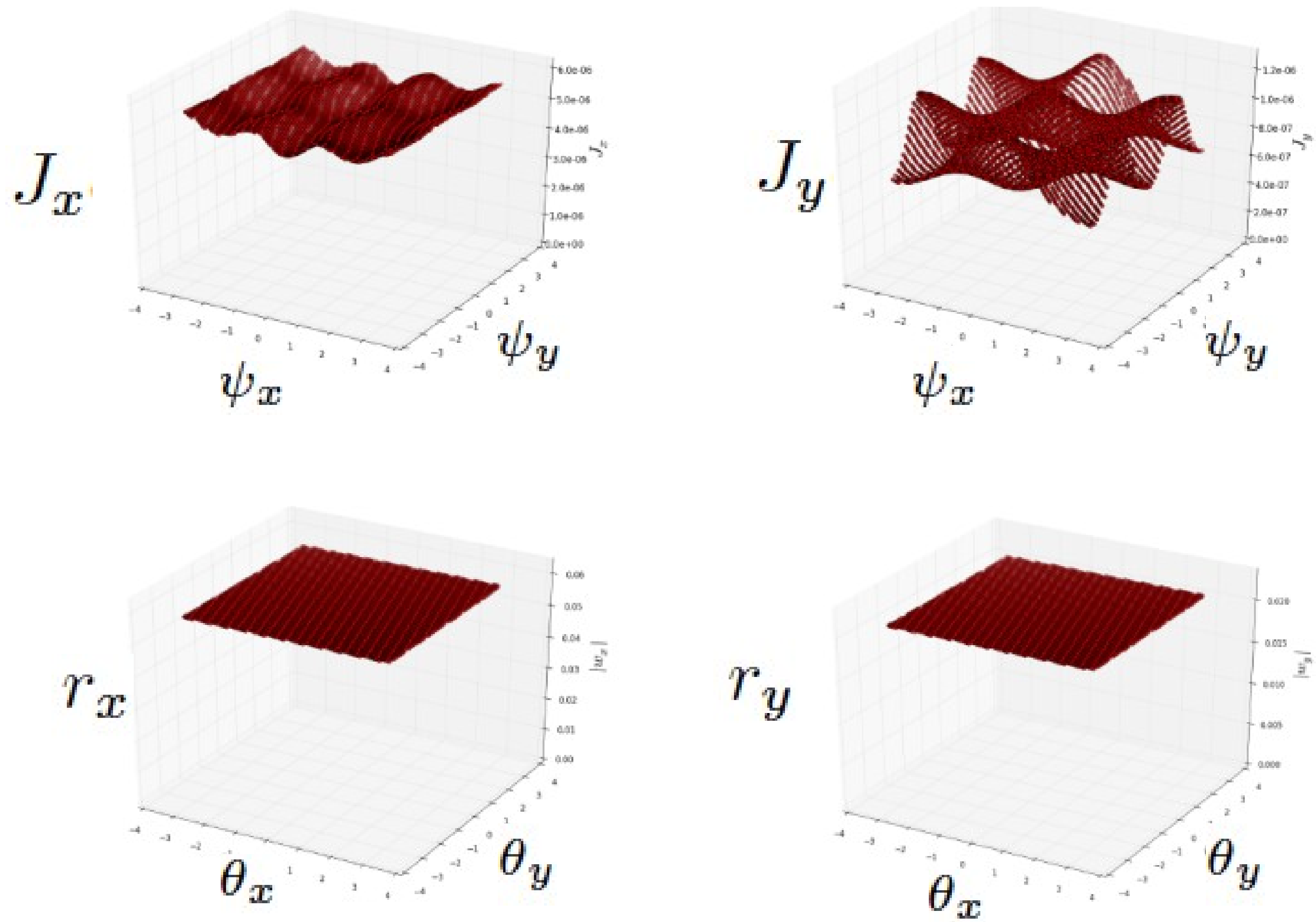
“Coherence condition”:

$$\mathrm{Im}\phi \approx 0; \Delta \approx 0.$$

Numerical Test, Poincare Sections: Strongly coupled x,y motion reduced to two simple independent rotations in separate planes

$$z_x = J_x e^{i\psi_x}, z_y = J_y e^{i\psi_y}$$

Courant-Snyder actions vary

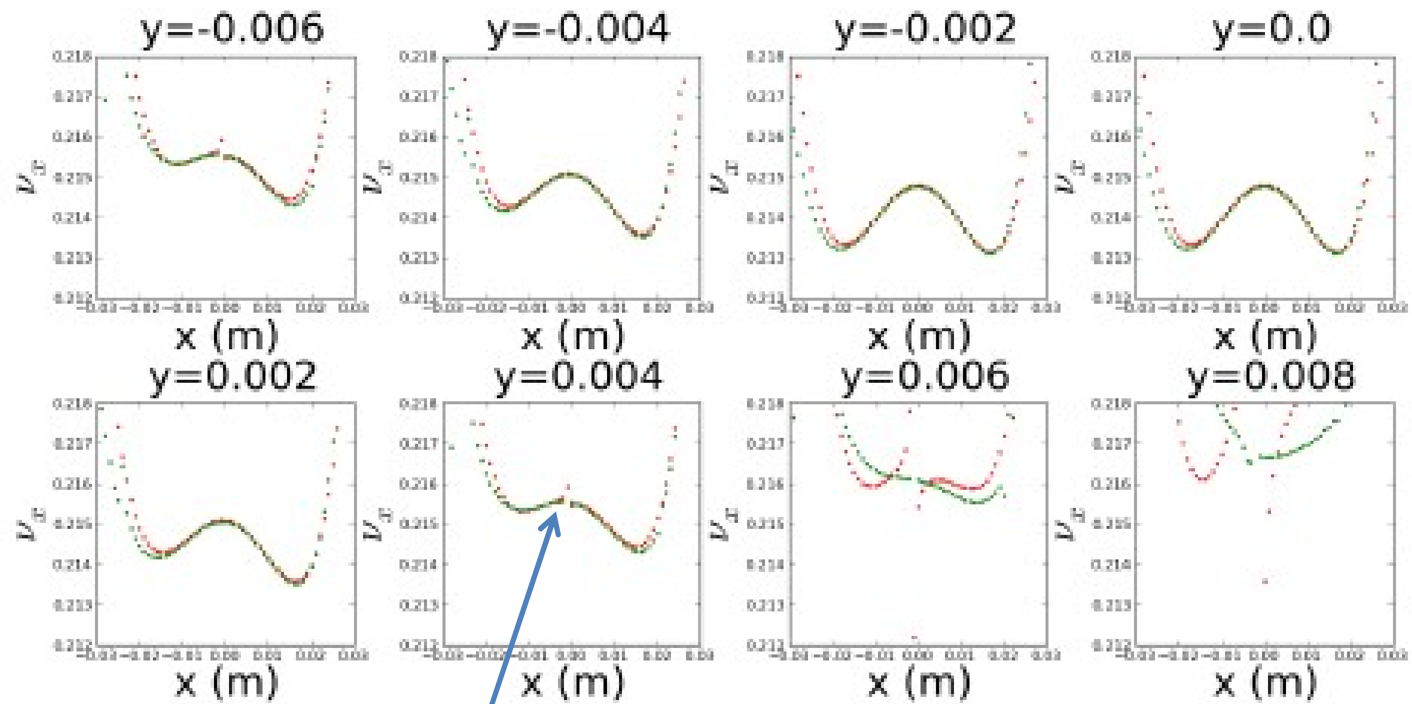


Initial x=10mm, y=2mm

$$w_x = r_x e^{i\theta_x}, w_y = r_y e^{i\theta_y}$$

New action-angle variables remain constant

Tune vz. Amplitude agrees with tracking



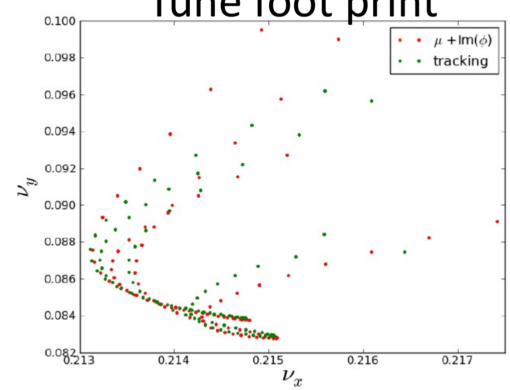
$$\nu_{x0} - \frac{\phi_x}{2\pi}$$

tracking

Resonance line revealed as jump in red curve $\nu_{x0} - \frac{\phi_x}{2\pi}$

And discontinuity in green (tracking)

Tune foot print

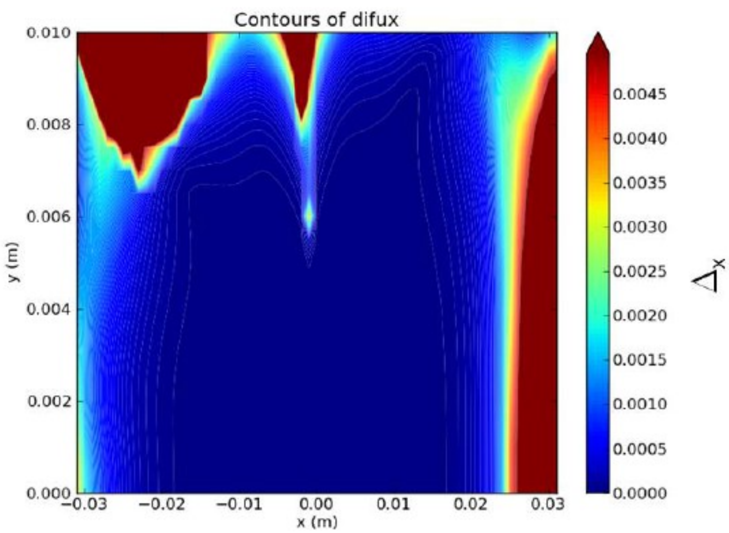


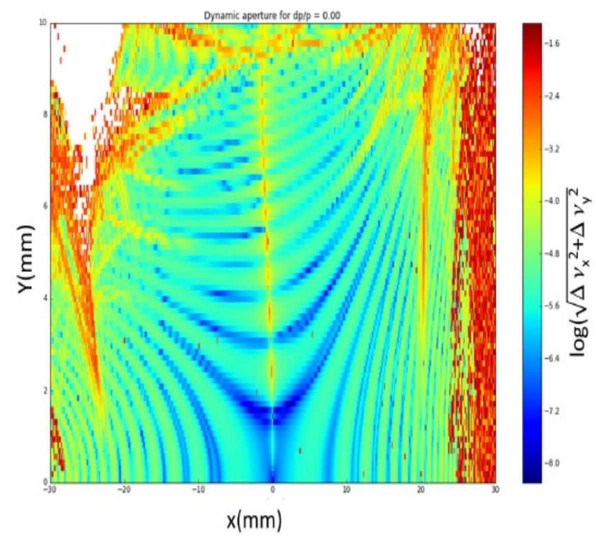
“Coherence condition”

$$\Delta \equiv \frac{w_2}{w_0} - (\frac{w_1}{w_0})^2 \approx 0$$

Gives frequency fluctuation, seems to be related to Liapunov exponent.

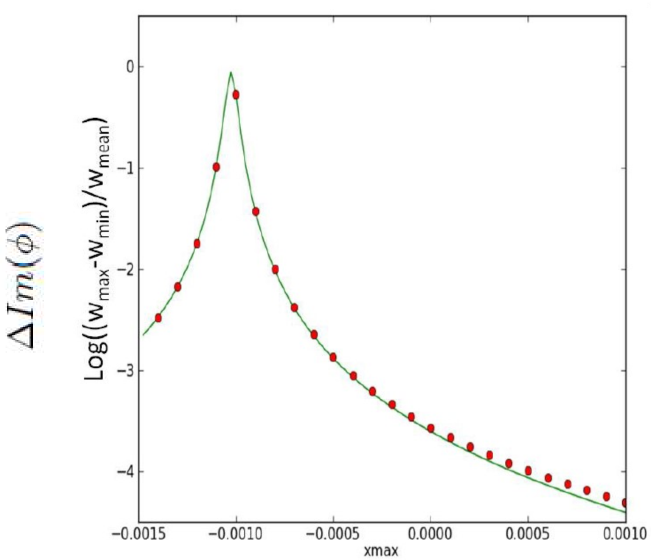
Δ_x Compared with → Frequency map, obtained by heavy tracking Calculation (Yongjun Li)

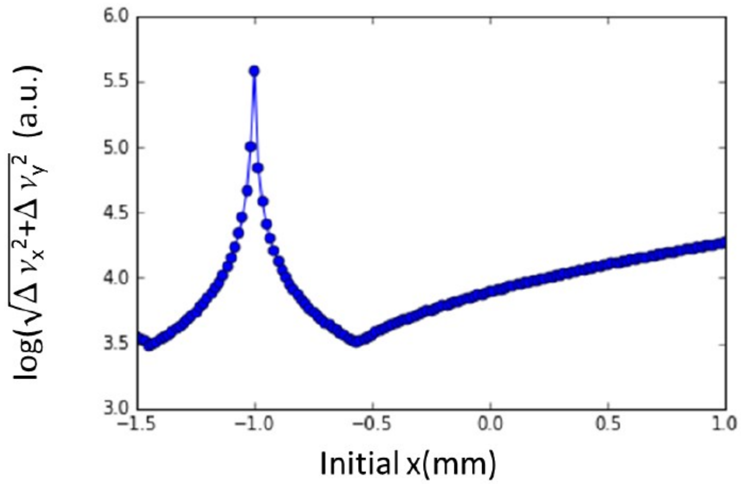




This can be used to optimize “dynamic aperture” of storage rings

Compare RMS of $\Delta w_x / w_x$ from tracking (red) with theory (green) times 4.1 around a resonance





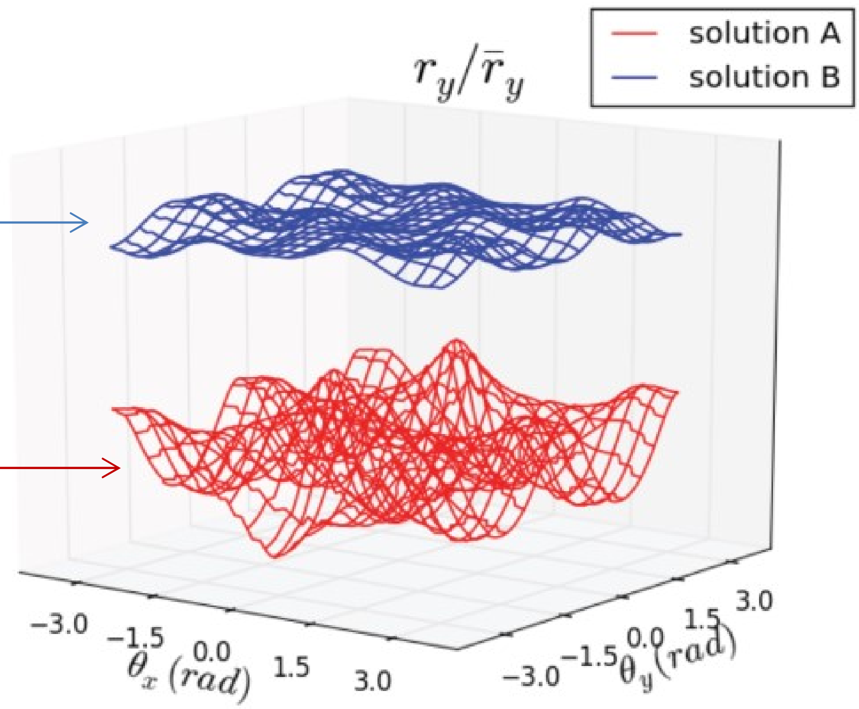
Scan x near resonance at x=-1mm y=6mm

Obtained from frequency map by Yongjun Li

Compare Poincare Sections of $r_y \equiv |w_y|$ for lattices optimized by nonlinear driving terms and by square matrix

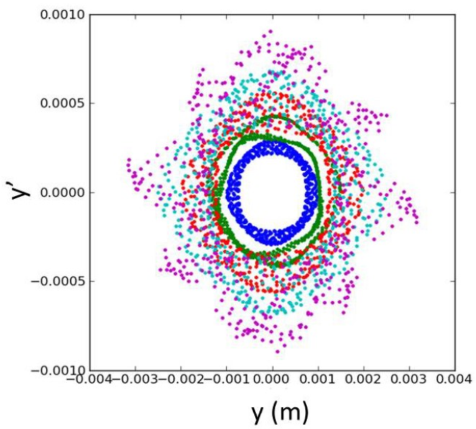
optimized by square matrix

optimized by nonlinear driving terms

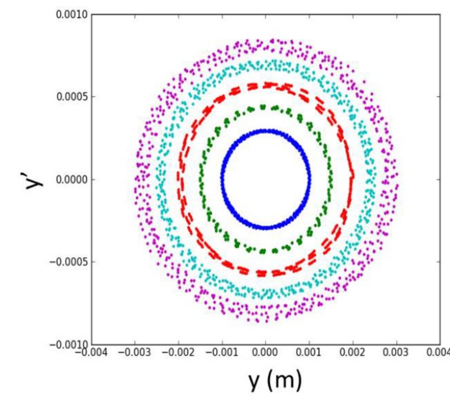


Phase space manipulation

5 particles with initial y increases proportional to initial x
Before and after minimization of $|\Delta w/w|$ by Yongjun Li



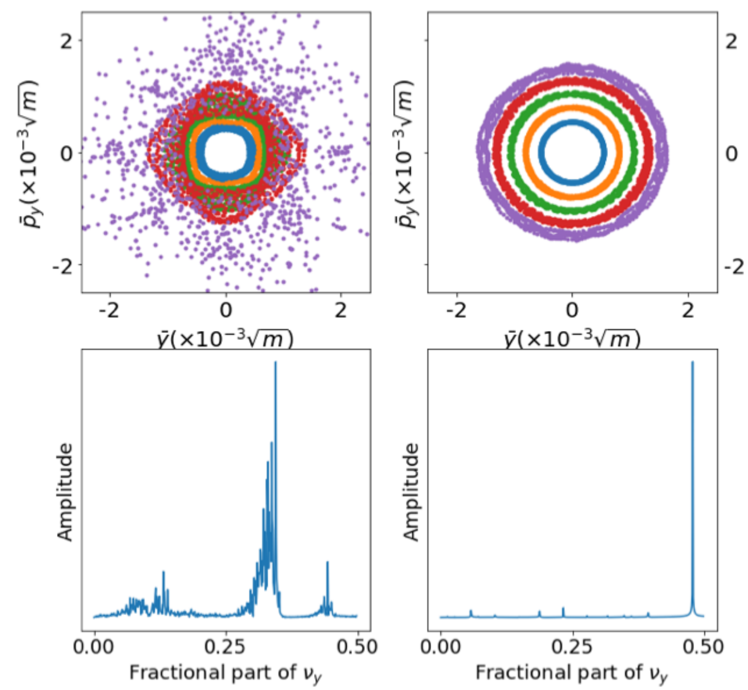
1000 turns



Conventional optimization

Square matrix optimization

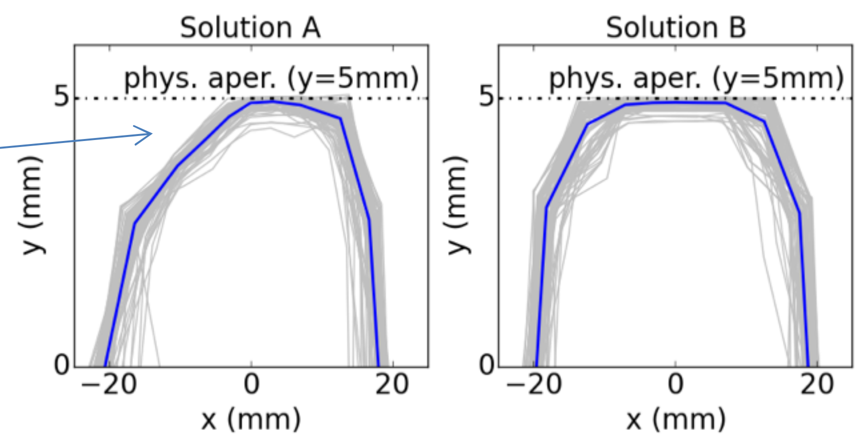
After 15000 turns



After 15000 turns
There are particles diffused
Into much larger y

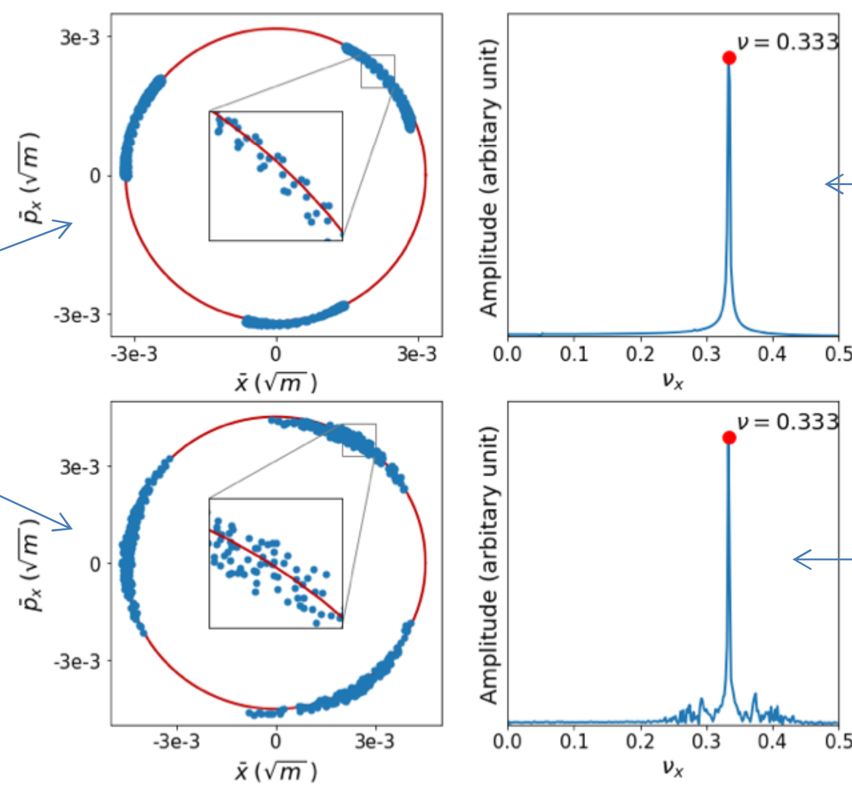
Spectrum is much more
Wide and complicated

Particles are lost in top left corner



Aperture is more symmetric with square matrix optimization

Square matrix optimized trajectory in phase space

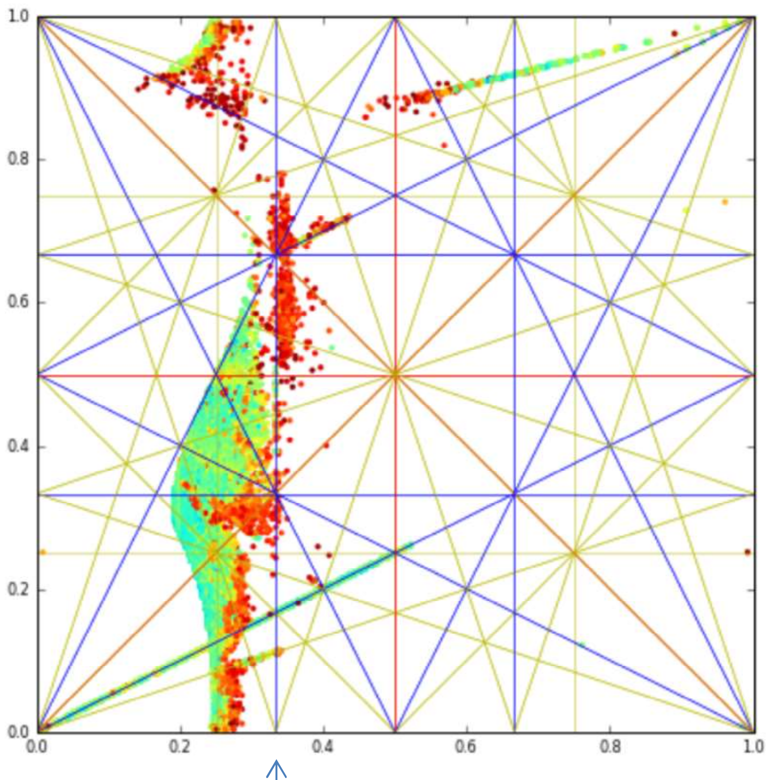


No magnet error

With magnet errors
The particles are still stable on 3rd order resonance

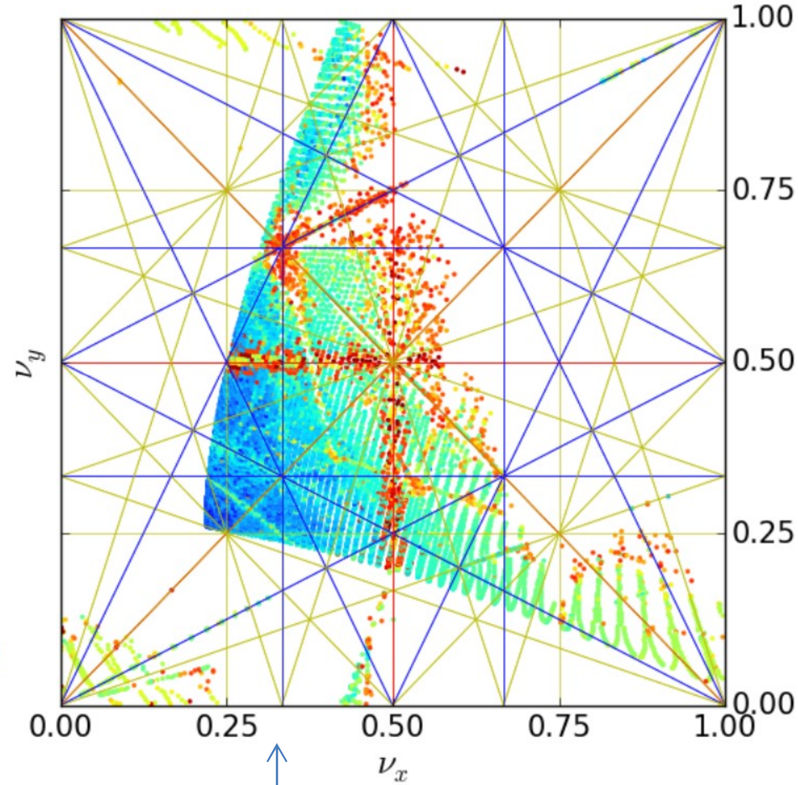
Tune footprint comparison of two approaches

Optimized by nonlinear driving terms



1/3 resonance line

Optimization obtained by square matrix



1/3 resonance line

Summary of off Resonance solution

- Square matrix $Z = MZ_0$
- One step to high order without iteration $UM = e^{i\mu I + \tau} U$
- Action-angle approximation $W \equiv UZ = \begin{bmatrix} w_0 \\ w_1 \\ \cdots \\ w_{m-1} \end{bmatrix}$

$$W = e^{i\mu I + \tau} W_0 \cong e^{i(\mu+\phi)} W_0.$$

- **Amplitude dependent tune** ϕ

Action $|w_0|$ **is nearly a constant:** $|\frac{\Delta W}{W}| \approx 0$

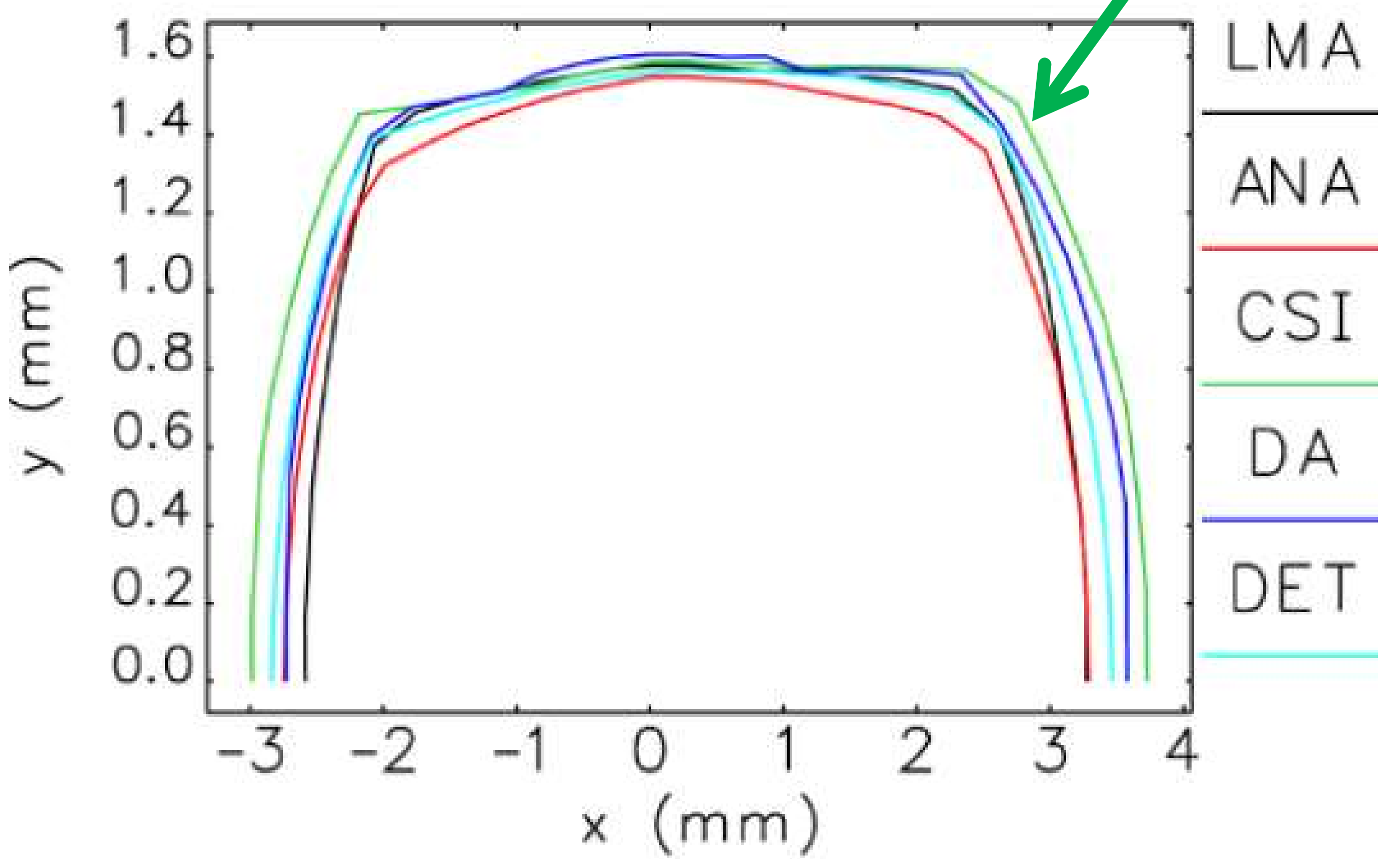
- **frequency fluctuation** $\Delta \equiv \frac{w_2}{w_0} - (\frac{w_1}{w_0})^2$ ≈0

amplitude fluctuation $|\frac{\Delta W}{W}|$

- "Coherence condition": $\mathrm{Im}\phi \approx 0; \Delta \approx 0.$ $|\frac{\Delta W}{W}| \approx 0$

DA obtained using various objectives

Based on concept developed from square matrix



Targets:

LMA: objective of dynamic acceptance, local momentum acceptance and chromatic detuning (as above)

ANA: objective of nonlinear chromaticity and driving/detuning terms

CSI: objective of CS invariant distortion and chromatic detuning, developed from the concept based on square matrix

DA: objective of on- and off-momentum dynamic acceptance, and chromatic detuning

DET: detuning of x-y grid (on and off momentum)

Yipeng Sun, Michael Borland
Argonne National Laboratory

High Brightness Synchrotron Light Source Workshop
April 26-28, 2017

A Celestial Dynamics Problem **Exactly on Resonance**: Henon-Heiles Problem

$$H = H_0 + H_1$$
$$H_0 = p_x^2 + p_y^2 + \frac{1}{2}(x^2 + y^2)$$
$$H_1 = x^2 y - \frac{y^3}{3}$$

$$V = W_x + aW_y$$
$$\dot{V} = (i\mu + \tau)W_x + a(i\mu + \tau)W_y \longleftarrow$$
$$\ddot{V} = (i\mu + \tau)^2 W_x + a(i\mu + \tau)^2 W_y$$
$$\cdots$$

Linear combination of two invariant spaces to find coherent solution

First rows of the matrixes give:

$$\begin{bmatrix} v \\ \dot{v} \\ \ddot{v} \\ \cdots \end{bmatrix} = \begin{bmatrix} w_{x0} & w_{y0} \\ i\mu\, w_{x0} + w_{x1} & i\mu\, w_{y0} + w_{y1} \\ (i\mu)^2 w_{x0} + 2i\mu\, w_{x1} + w_{x2} & (i\mu)^2 w_{y0} + 2i\mu\, w_{y1} + w_{y2} \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix}$$

Coherence condition

$$\dot{v} = \lambda v$$
$$\ddot{v} = \lambda \dot{v}$$
$$\cdots$$

$$\begin{bmatrix} i\mu\, w_{x0} + w_{x1} & i\mu\, w_{y0} + w_{y1} \\ (i\mu)^2 w_{x0} + 2i\mu\, w_{x1} + w_{x2} & (i\mu)^2 w_{x0} + 2i\mu\, w_{x1} + w_{x2} \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix} = \lambda \begin{bmatrix} w_{x0} & w_{y0} \\ i\mu\, w_{x0} + w_{x1} & i\mu\, w_{y0} + w_{y1} \\ \cdots & \cdots \end{bmatrix} \begin{bmatrix} 1 \\ a \end{bmatrix}$$

Eigenvalue Equation

$$\begin{bmatrix} i\mu\, w_{x0} + w_{x1} - \lambda w_{x0} & i\mu\, w_{y0} + w_{y1} - \lambda w_{y0} \\ i\mu\, w_{x1} + w_{x2} - \lambda w_{x1} & i\mu\, w_{y1} + w_{y2} - \lambda w_{y1} \end{bmatrix} X = 0 \qquad X \equiv \begin{bmatrix} 1 \\ a \end{bmatrix}$$

Let $i\phi = \lambda - i\mu$,

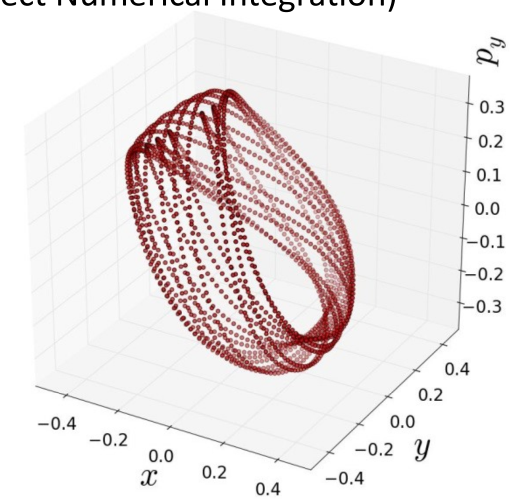
$$\begin{bmatrix} w_{x0} & w_{y0} \\ w_{x1} & w_{y1} \end{bmatrix}^{-1} \begin{bmatrix} w_{x1} & w_{y1} \\ w_{x2} & w_{y2} \end{bmatrix} X = i\phi X \longleftarrow$$

A generalization of frequency shift $\phi = -i\dfrac{w_1}{w_0}$

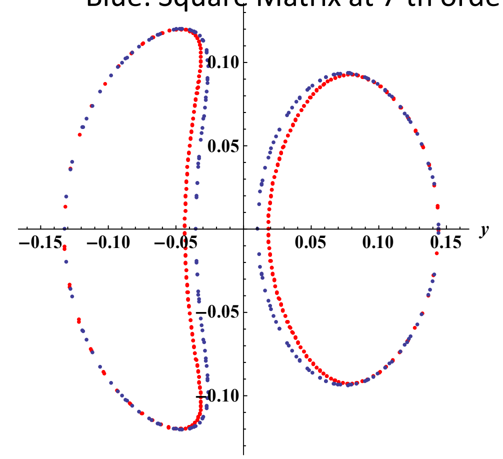
$i\phi$ is a solution of a quadratic equation, there are two solutions v_1, v_2

Solution on Henon-Heiles Problem: Exactly on Resonance

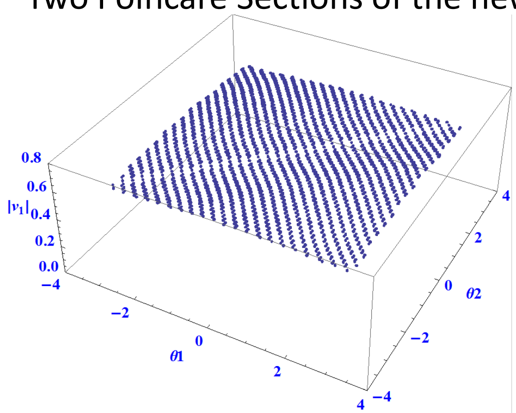
Poincare Section
(Direct Numerical Integration)

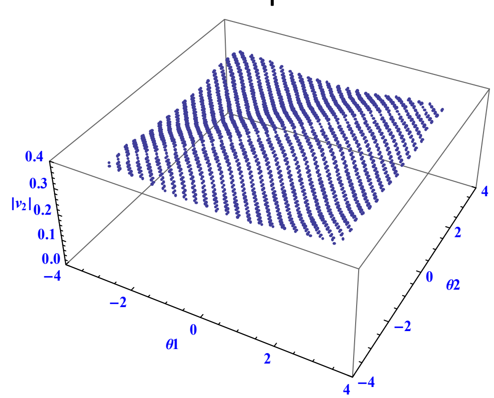


Cross section at x=0
Red: Direct Numerical Integration
Blue: Square Matrix at 7’th order



Two Poincare Sections of the new actions show two independent rotations





A way to avoid small denominator problem?

- Clearly, this method is general, and valid for more than two frequencies in resonance.
- Hence this method provides a way to surround the small denominator problem